

INVERTIBILITY OF TOEPLITZ OPERATORS WITH POLYANALYTIC SYMBOLS

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ABSTRACT. For a class of continuous functions including complex polynomials in z, \bar{z} , we show that the corresponding Toeplitz operator on the Bergman space of the unit disc can be expressed as a quotient of certain differential operators with holomorphic coefficients. This enables us to obtain several operator theoretic results including a criterion for invertibility of a Toeplitz operator using theory of linear analytic ordinary differential equations.

Throughout $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ will denote the unit disc and $A^2(\mathbb{D})$ will denote the Bergman space of square integrable holomorphic functions on \mathbb{D} with respect to the normalized Lebesgue measure. Also, by $H(\mathbb{D})$ will denote the Frechet space of all holomorphic functions on \mathbb{D} , while $H(\bar{\mathbb{D}})$ denotes the set of all holomorphic functions defined on a neighbourhood of $\bar{\mathbb{D}}$. given a bounded measurable function $g \in L^\infty(\mathbb{D})$, we will denote by $T_g : A^2(\mathbb{D}) \rightarrow A^2(\mathbb{D})$ the corresponding Toeplitz operator. Recall its definition:

$$T_g(f) = P(gf), \quad f \in A^2(\mathbb{D}),$$

where $P : L^2(\mathbb{D}) \rightarrow A^2(\mathbb{D})$ denotes the orthogonal projection. Let $\phi \in H(\bar{\mathbb{D}})[\bar{z}]$ be a polyanalytic function on $\bar{\mathbb{D}}$. We are interested in the question of invertibility of the corresponding Toeplitz operator $T_\phi : A^2(\mathbb{D}) \rightarrow A^2(\mathbb{D})$. More generally we are interested in determining dimensions of its kernel and cokernel. Similar problem in the setting of the Hardy space $H^2(\mathbb{D})$ is well understood. Indeed, the following is well-known.

Theorem 0.1. ([D]). *Let $g \in C(\bar{\mathbb{D}})$. Then the corresponding Toeplitz operator $T_g : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ is invertible if and only if $g|_{\partial\mathbb{D}}$ does not vanish and the winding number of $g(\partial\mathbb{D})$ around 0 is 0.*

We recall the following well-known partial analogue of this statement in the Bergman space setting (see for example [[SZ1], Theorem 24].)

Lemma 0.1. *If $g \in C(\bar{\mathbb{D}})$, then $T_g : A^2(\mathbb{D}) \rightarrow A^2(\mathbb{D})$ is a Fredholm operator if and only if g does not vanish on $\partial\mathbb{D}$, in this case its index equals to the negative of the winding number of $g(\partial\mathbb{D})$ around 0.*

The full analogue of Theorem 0.1 in the Bergman space setting fails even for harmonic functions: Sundberg and Zheng [[SZ], Theorem 2.3] constructed an example $g = \bar{z} + \psi, \psi \in H(\bar{\mathbb{D}})$ such that T_g is not invertible

while the winding number of $g(\partial\mathbb{D})$ around 0 is 0. Moreover, they showed that 0 is the isolated point of the spectrum of T_g .

Recall the following well known formula

$$T_{\bar{z}^k}(z^n) = \begin{cases} 0 & k > n, \\ \frac{n-k+1}{n+1} z^{n-k} & n \geq k. \end{cases}$$

This easily implies that for any polynomial $f \in \mathbb{C}[z]$, we have

$$T_{\bar{z}^k}(f) = \prod_{i=2}^{k+1} (zD + i)^{-1} D^k(f),$$

Where $D : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ denotes the differentiation operator, and

$$zD + i : H(\mathbb{D}) \rightarrow H(\mathbb{D}), \quad i > 0$$

are understood as invertible differential operators.

In particular, $T_{\bar{z}} = (zD + 2)^{-1} D$ which is equivalent to the following formula from Sundberg-Zheng [SZ]

$$T_{\bar{z}} f = \frac{1}{z^2} \int_0^z w f'(w) dw.$$

The following is the key

Lemma 0.2. *Let $f, g \in A^2(\mathbb{D})$. Let $\phi(z) = \sum_{i=0}^n a_i(z) \bar{z}^i$, $a_i(z) \in H^\infty(D)$ be an n -th order polyanalytic function on \mathbb{D} . Then $T_\phi(f) = g$ if and only if*

$$\prod_{i=2}^{n+1} (zD + i)(g) = \prod_{i=2}^{n+1} (zD + i)(a_0(z)f) + \sum_{i=1}^n \prod_{k=i+2}^{n+1} (zD + k) D^i a_i(z)(f)$$

Proof. Since $\Lambda = \prod_{i=2}^{n+1} (zD + i) : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ is an injective linear operator, it suffices to check that $\Lambda(T_\phi)(f) = \Lambda(g)$. Hence we need to show that

$$\Lambda(T_{a_i(z)\bar{z}^i})(f) = \prod_{k=i+2}^{n+1} (zD + k) D^i (a_i(z)f)$$

It suffices to check this equality for $f = z^m, m \geq 0$. But this is immediate from the above discussion. □

It will be convenient to introduce the following notations. Given an n -th order polyanalytic function $g(z) = \sum_{i=0}^n g_i(z) \bar{z}^i, g_i \in H(\mathbb{D})$, we will put

$$\hat{g}(z) = \sum_{i=0}^n g_i(z) z^{-i}, \quad \tilde{g}(z) = \sum_{i=0}^n g_i(z) z^{n-i} = z^n \hat{g}(z).$$

Thus $g(z) = \hat{g}(z)|_{\partial\mathbb{D}}$.

From now we will fix an n -th order polyanalytic function $\phi = \sum_{i=0}^n a_i(z) \bar{z}^i$ where $a_i(z) \in H(\mathbb{D})$. Motivated by Lemma 0.2, we will denote the following n -th order differential operator by D_ϕ :

$$D_\phi = \prod_{i=2}^{n+1} (zD + i) a_0(z) + \sum_{i=1}^n \prod_{k=i+2}^{n+1} (zD + k) D^i a_i(z).$$

It follows from the above lemma that $f \in \text{Ker} T_\phi$ if and only if $f \in A^2(\mathbb{D})$ is a solution to the following n -th order linear homogeneous differential equation

$$D_\phi(f) = 0.$$

Next we will compute the first two leading terms of D_ϕ , i.e. coefficients in front of D^n, D^{n-1} . Clearly the leading term of D_ϕ is

$$\tilde{\phi} D^n = \sum_{i=0}^n a_i(z) z^{n-i} D^n = z^n \left(\sum_{i=0}^n a_i(z) z^{-i} \right) D^n = z^n \hat{\phi} D^n.$$

Recall that the following commutator relation holds in the ring of differential operators

$$Dg(z) - g(z)D = g'(z), \quad g(z) \in H(D).$$

Using this relation we easily obtain the following expansion in terms of powers of D

$$\prod_{k=1}^m (zD + b_k) = z^m D^m + (m(m-1)/2 + \sum b_k) z^{m-1} D^{m-1} + \dots, b_k \in H(\mathbb{D}).$$

Thus

$$\prod_{k=1}^m (zD + b_k) \psi = \psi z^m D^m + ((m(m-1)/2 + \sum b_k) z^{m-1} \psi + m z^m \psi') D^{m-1} + \dots, \psi \in H(\mathbb{D}).$$

Now our differential operator is D_ϕ is

$$\prod_{i=2}^{n+1} (zD + i) \phi + D^n a_n(z) + (zD + n+1) D^{n-1} a_{n-1}(z) + \dots + \prod_{i=3}^{n+1} (zD + i) \partial a_1(z).$$

Therefore the coefficient in front of D^{n-1} is

$$n(n+1)a_0(z)z^{n-1} + n z^n a_0'(z) + n a_n'(z) + (a_{n-1}(z) \sum_{k < n-1} (n+1)(k+1) + n a_{n-k}'(z)) z^k.$$

Which may be written as

$$(n+1) \left(\sum_{k=0}^n a_k z^{n-k} \right)' + n \left(\sum_{k=0}^n a_k' z^{n-k} \right),$$

therefore it is equal to

$$(n+1)\tilde{\phi}' - \sum a_k' z^{n-k} = (n+1)\tilde{\phi}' - \frac{\partial \tilde{\phi}}{\partial z}.$$

To summarize we have

$$D_\phi = \tilde{\phi} D^n + ((n+1)\tilde{\phi}' - \frac{\partial \tilde{\phi}}{\partial z}) D^{n-1} + \dots$$

From now on we will assume that ϕ does not vanish on $\partial\mathbb{D}$. Recall that it follows from the argument principle that if the winding number of $\phi(\partial\mathbb{D})$ around 0 is m , then $\tilde{\phi}$ has $n - m$ zeros on \mathbb{D} . We have the following

Theorem 0.2. *Assume that $\tilde{\phi}$ has one zero w on \mathbb{D} such that*

$$\text{res}_w(\frac{\partial \tilde{\phi}}{\partial z} \tilde{\phi}^{-1}) \notin n+1 + \mathbb{Z}_{\geq 0}.$$

Then the Toeplitz operator T_ϕ is onto with $n - 1$ -dimensional kernel.

Proof. Turning the n -th order differential equation $D_\phi f = 0$ into a system of ordinary differential equations in the usual way $y' = Ay$, it follows from our assumptions that $\text{res}_w A$ has eigenvalue 0 with multiplicity $n - 1$ and no other eigenvalue which is a positive integer. Thus by a standard result in ordinary differential equations (see for example [[H], Theorem 11.4]), there are $n - 1$ linearly independent holomorphic solutions around w . Now since the index of T_ϕ is $n - 1$, this implies our assertion. \square

Remark 0.1. Let ϕ as above be such that the winding number around 0 of $\phi(\partial\mathbb{D})$ is 0. Then for such generic ϕ , the corresponding Toeplitz operator is invertible. Indeed, let w_1, \dots, w_n be zeros of $\tilde{\phi}$ on \mathbb{D} . It follows that T_ϕ is not invertible if and only if equation $D_\phi f = 0$ has a nontrivial solution in $A^2(\mathbb{D})$. Let $y' = Ay$ be the matrix form of this equation as in proof of Theorem 0.2. For generic such ϕ it follows that this equation has regular singularities at w_i . Put $A_i = \text{res}_{w_i} A$. Thus generically, distinct eigenvalues of A_i do not differ by integers. Let M_1, \dots, M_n be the monodromy matrices around w_1, \dots, w_n respectively. Then M_i is conjugate to $\exp(2\pi i A_i)$, hence it has an eigenvalue 1 with multiplicity $n - 1$. Then existence of such a solution implies that matrices M_1, \dots, M_n have a simultaneous eigenvector with eigenvalue 1. But generically this does not hold.

Next we will illustrate the case of $n = 2$. For simplicity let us assume that $\phi = a_2 \bar{z}^2 + a_1 \bar{z} + a_0(z)$, $a_1, a_2 \in \mathbb{C}$. Then the corresponding differential equation $D_\phi f = 0$ becomes

$$(6 + z^2 D^2 + 6zD)(a_0(z)f) + a_2 D^2 f + a_1(zD + 3)Df = 0,$$

which may be rewritten as

$$h(z)f'' + ((3h' - z^2 a_0'(z))f' + (6h'' + 6za_0'(z) + z^2 a_0''(z))f = 0.$$

Then we have the following

Theorem 0.3. *Let $0 \neq w \in \mathbb{D}$, and let $g \in H(\bar{\mathbb{D}})$ be such that $-\frac{1}{w^2} \notin z^2 g(z)(\bar{\mathbb{D}})$. Put $\phi = (z - w)^2 g(z) + (\bar{z} - w^{-1})^2$. Then the Toeplitz operator T_ϕ is invertible if the following equation has no roots in \mathbb{Z}_+ .*

$$\lambda(\lambda - 1) + \left(3 - \frac{2w^2}{w^{-2} + w^2g(w)}\right)\lambda + \frac{6w^{-2} + 2w^2g(w)}{w^{-2} + w^2g(w)} = 0.$$

Proof. It follows that $\tilde{\phi}$ has exactly one zero in \mathbb{D} , hence T_ϕ has index 0. Thus T_ϕ is invertible if the differential equation $D_\phi f = 0$ has no nontrivial solutions in $A^2(\mathbb{D})$. Now the above equation is the indicial equation of our second order differential equation $D_\phi f = 0$ around w . Thus the equation having no solution in \mathbb{Z}_+ implies that there are no holomorphic solutions around w . Thus T_ϕ is injective, and since it has index 0 it must be bijective. \square

Remark 0.2. One can formulate similar statement about invertibility of Toeplitz operators of harmonic functions of the form

$$\phi(z) = (z - w)^n g(z) + (\bar{z} - w^{-1})^n,$$

where $w^{-n} \notin z^n g(z)(\bar{\mathbb{D}})$. Then T_ϕ is invertible unless the corresponding indicial equation has a root in \mathbb{Z}_+ .

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